

# SECONDARY UPSILON INVARIANTS OF KNOTS

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**ABSTRACT.** The knot invariant Upsilon, defined by Ozsváth, Stipsicz, and Szabó, induces a homomorphism from the smooth knot concordance group to the group of piecewise linear functions on the interval  $[0, 2]$ . Here we define a set of related secondary invariants, each of which assigns to a knot a piecewise linear function on  $[0, 2]$ . These secondary invariants provide bounds on the genus and concordance genus of knots. Examples of knots for which Upsilon vanishes but which are detected by these secondary invariants are presented.

## 1. INTRODUCTION

In [14], Ozsváth, Stipsicz, and Szabó defined the *Upsilon* invariant of knots  $K \subset S^3$ , denoted  $\Upsilon_K(t)$ . The map  $K \rightarrow \Upsilon_K$  is a homomorphism from the smooth knot concordance group to the group of piecewise linear functions on the interval  $[0, 2]$ . This homomorphism provides bounds on the three-genus, the four-genus, and the concordance genus of knots. Since then, this invariant has been used to effectively address a range of problems; for a sampling, see [1, 2, 3, 4, 11, 13, 15, 16].

For each  $t \in (0, 2)$ , we will define a secondary function denoted  $\Upsilon_{K,t}^2(s)$ . This is again a piecewise linear function on the interval  $[0, 2]$  and is a concordance invariant that provides bounds on the three-genus and concordance genus of knots. For any piecewise linear function  $f(t)$  on  $[0, 2]$ , the jump function of the derivative,  $\Delta f'(t)$ , is well-defined for all  $t \in (0, 2)$ . If  $\Delta \Upsilon'_K(t) > 0$ , then  $\Upsilon_{K,t}^2(s) < 0$  for all  $s$ . The most interesting case is when  $\Delta \Upsilon'_K(t) = 0$  for all  $t$  (that is, when  $\Upsilon_K(t)$  is identically 0) but  $\Upsilon_{K,t}^2(s)$  is a nontrivial function for some values of  $t$ .

Jen Hom [10] constructed a knot  $K$  with vanishing  $\Upsilon$ -invariant that can be shown not to be slice using the  $\epsilon$ -invariant defined in [8]. We will show that  $K$  can also be quickly shown to be nontrivial using  $\Upsilon^2$ . As an added feature, a corollary of the genus bounds imposed by  $\Upsilon^2$  is that  $g_c(nK) \geq 4n - 2$ . In particular, this yields example for which  $g_c$  can be arbitrarily large but have vanishing  $\Upsilon$ .

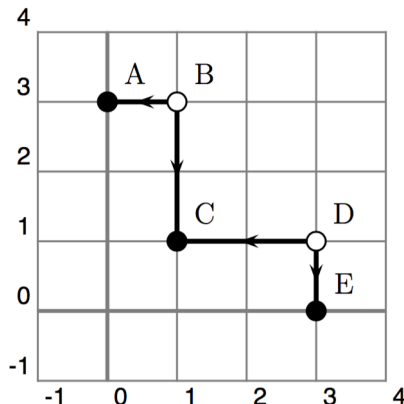
As a final set of examples, we describe an infinite set of complexes for which both  $\Upsilon$  and  $\epsilon$  vanish, but which can be shown to be independent using  $\Upsilon^2$ ; whether these complexes arise from actual knots has not been determined.

## 2. KNOT COMPLEXES, $\text{CFK}^\infty(K)$

To each knot  $K \subset S^3$ , there is an associated bifiltered graded chain complex  $\text{CFK}^\infty(K)$  which has a compatible structure as a  $\mathbb{F}[U, U^{-1}]$ -module, where  $\mathbb{F}$  is the field with two elements. We abbreviate  $\mathbb{F}[U, U^{-1}]$  by  $\Lambda$ . As we explain in this section, Figure 1 provides a schematic illustration of the complex  $\text{CFK}^\infty(K)$  associated to the torus knot  $K = T_{3,4}$ .

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FIGURE 1.  $\text{CFK}^{\infty}(T(3, 4))$ 

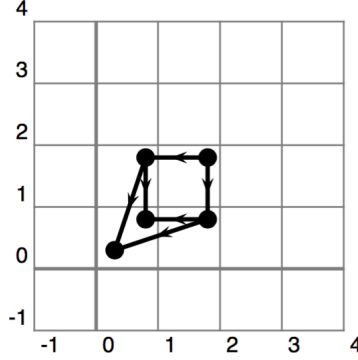
We now write  $C$  for  $\text{CFK}^{\infty}(K)$ . In our example, there are three black dots in the diagram; these represent basis elements for the set of elements of grading 0,  $C_0$ . The two white dots represent a basis for  $C_1$ . The arrows represent the boundary map: for instance,  $\partial C = 0$  and  $\partial B = A + C$ . The complex is bifiltered. The coordinates  $(i, j)$  of a vertex represent the *algebraic* and *Alexander* filtration levels of each of the generators. For any element in the complex, we write  $\text{alg}(x)$  and  $\text{Alex}(x)$  for the two filtration levels.

The complex as drawn represents, in the notation of [7], a model complex for  $C$  in which the  $\Lambda$ -structure is hidden. The full complex is formed by taking all integer diagonal translates of the illustrated complex; the action of  $U$  shifts the vertices a distance of one down and to the left. More formally, the diagram illustrates a finite dimensional complex  $C'$  and  $C = C' \otimes \Lambda$ , graded and filtered so that the action of  $U$  lowers gradings by 2 and filtration levels by 1.

All the knot complexes we consider can be described by schematic diagrams of such model complexes. In each case, the model  $C'$  has the property that its homology is  $H(C') \cong \mathbb{F}$ , with the generator in grading 0. Furthermore, the minimal algebraic filtration level of representatives of the nontrivial homology class is 0; similarly for the Alexander filtration level. Thus, if the diagram is connected, it determines the grading level of all vertices.

Typically, there can be more than one vertex at a given bifiltration level. Since we cannot illustrate more than one vertex at a given lattice point  $(i, j)$ , we will place such vertices in the unit square with bottom left corner at  $(i, j)$ . Figure 2 illustrates a model complex  $C$  for the Figure 8 knot,  $4_1$ . The two vertices at bifiltration level  $(0, 0)$  are both cycles representing a generator of  $H_0(C)$ .

**2.1. Formal knot complexes.** The essential features of  $C = \text{CFK}^{\infty}(K)$  are as follows. It is a doubly filtered graded complex with a compatible  $\Lambda$ -module structure. As a  $\Lambda$ -module,  $C$  is finitely generated and free. The action of  $U \in \Lambda$  lowers filtration levels by one and gradings by two. The homology is given by  $H(C) \cong \Lambda$  with  $1 \in \Lambda$  representing a generator of  $H_0(C)$ . Finally, there is a symmetry property: the complex formed by switching the two filtrations is chain homotopy equivalent to  $C$  by a filtered graded chain homotopy equivalence.

FIGURE 2. The figure eight knot:  $\text{CFK}^\infty(4_1)$ 

All of our work applies to any complex with these properties. We will refer to them as  $\mathcal{K}$ -complexes.

**2.2. Concordance of complexes.** The following result is proved by Hom in [9].

**Proposition 2.1.** *If  $K$  and  $J$  are concordant knots, then there are acyclic complexes  $A_1$  and  $A_2$  such that  $\text{CFK}^\infty(K) \oplus A_1 = \text{CFK}^\infty(J) \oplus A_2$ .*

Here the  $A_i$  are acyclic as graded complexes but not necessarily as filtered complexes. We define two  $\mathcal{K}$ -complexes to be *concordant* if they are similarly stably equivalent. A standard argument shows that the set of concordance classes forms an abelian group under tensor products.

### 3. UPSILON, $\Upsilon_K$

In defining  $\Upsilon_K$ , we follow the presentation in [12]. For any  $t \in [0, 2]$  and  $s \in \mathbb{R}$ , we define the subcomplex  $\mathcal{F}_{t,s} \subset \text{CFK}^\infty(K)$  as follows. Let  $\mathcal{B}$  denote a bifiltered graded basis of  $\text{CFK}^\infty(K)$ .

$$\mathcal{F}_{t,s} = \langle \{x \in \mathcal{B} \mid (t/2)\text{Alex}(x) + (1 - t/2)\text{Alg}(x) \leq s\} \rangle.$$

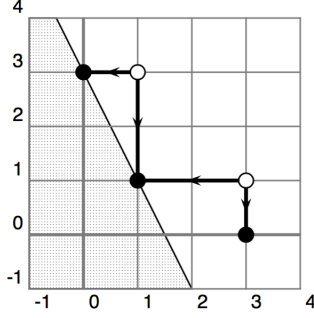
This is independent of the choice of bifiltered graded basis.

In Figure 3 we illustrate the complex  $\text{CFK}^\infty(T_{3,4})$  with a half-space shaded in. This shaded region represents the subcomplex  $\mathcal{F}_{2/3,1}$ . In general,  $\mathcal{F}_{t,s}$  is represented by a half-space with boundary a line of slope  $m = 1 - 2/t$ ; a half-space with boundary of slope  $m$  corresponds to  $\mathcal{F}_{t,s}$  with  $t = 2/(1 - m)$ . Given the bounding line, the value of  $s$  is given by  $tj_0/2$ , where  $j_0$  is the  $j$ -intercept of the line. We call the bounding line for the half-space the *support line* for  $\mathcal{F}_{t,s}$ , which we denote  $\mathcal{L}_{t,s}$ .

Given this, we can now define  $\Upsilon_K(t)$  for any  $t \in [0, 2]$ .

**Definition 3.1.** Let

$$\gamma_K(t) = \min\{s \mid H_0(\mathcal{F}_{t,s}) \rightarrow H_0(\text{CFK}^\infty(K)) \cong \mathbb{F} \text{ is surjective}\}.$$

FIGURE 3.  $\text{CFK}^\infty(T(3,4))$  doubly filtered

Define

$$\Upsilon_K(t) = -2\gamma_K(t).$$

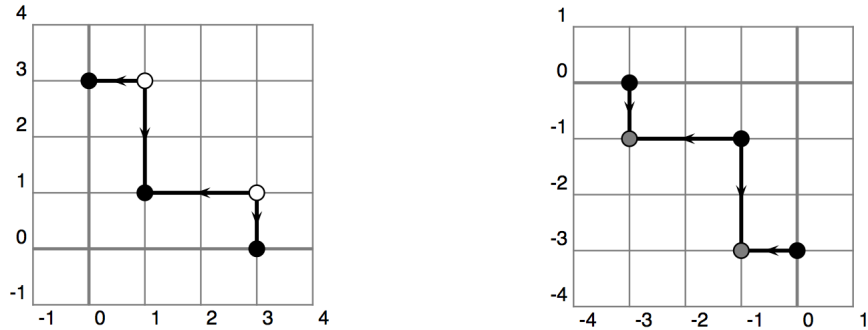
That  $\Upsilon_K$  is well defined follows readily from Proposition 2.1.

**Corollary 3.2.** *If  $K$  and  $J$  are concordant, then  $\Upsilon_K = \Upsilon_J$ .*

**3.1. Vertices and pivot points.** Let  $\mathcal{P}$  denote the set of bifiltration levels of elements of  $\text{CFK}^\infty(K)$ . (This depends on the homotopy representative of the complex.) Fix a value of  $t \in [0, 2]$  and recall that  $\gamma_K(t) = -\Upsilon_K(t)/2$ . The line  $\mathcal{L}_{t,\gamma(t)}$  contains a nonempty subset of  $\mathcal{P}$  which we denote  $\mathcal{P}_t$ . In the illustrated example, the subset is  $\mathcal{P}_{2/3} = \{(0, 3), (1, 1)\}$ . For small values of  $\delta$ ,  $\mathcal{P}_{t-\delta}$  contains exactly one element of  $\mathcal{P}_t$  and  $\mathcal{P}_{t+\delta}$  also contains exactly one element of  $\mathcal{P}_t$ . We call these the *negative* and *positive pivot points* at  $t$ ,  $p_t^-$  and  $p_t^+$ . The next result is proved in [12], although the statement there places the difference in absolute value.

**Theorem 3.3.** *The function  $\Upsilon_K(t)$  has a singularity at  $t$  if and only if  $p_t^- \neq p_t^+$ . In general, for  $t \in (0, 2)$ ,  $\Delta\Upsilon'_k(t) = \frac{2}{t} (i(p_t^+) - i(p_t^-))$  where  $i(p)$  denotes the first coordinate of a lattice point  $p$ .*

**Example 3.4.** As an example, Figure 4 illustrates the complexes for the torus knots  $T(3, 4)$  and  $-T(3, 4)$ . Black dots are at grading level 0, white dots at grading 1, and gray dots at grading  $-1$ . (The entire complexes include all the  $U^k$  translates of these diagrams.)

FIGURE 4.  $\text{CFK}^\infty(T(3,4))$  and  $\text{CFK}^\infty(-T(3,4))$

For  $T(3, 4)$ , each of the black dots represents a generator of  $H_0(\text{CFK}^\infty(K))$ . For lines of slopes between  $-\infty$  and  $-2$ , that is, for  $t \in [0, 2/3]$ , those that go through the vertex  $(0, 3)$  contain one of the generators, and no line of lesser  $y$ -intercept does. Thus, for  $t$  in this interval, we have

$$\Upsilon_{T(3,4)}(t) = -2(3t/2) = -3t.$$

For  $t \in [2/3, 1]$ , the line of slope  $m = 1 - 2/t$  that contains one of the generators goes through the vertex at  $(1, 1)$ . This line has  $y$ -intercept  $y_0 = 1 - m = 2/t$ , and thus

$$\Upsilon_{T(3,4)}(t) = -2(y_0 t/2) = -2(2/2) = -2.$$

**Example 3.5.** For the knot  $-T(3, 4)$ , the only cycle representing a generator of  $H_0(\text{CFK}^\infty(K))$  is the sum of the generators at bifiltration levels  $(-3, 0)$ ,  $(-1, -1)$  and  $(0, -3)$ . If  $t \in [0, 2/3]$  (that is, if the corresponding line has slope less than or equal to  $-2$ ), then for  $\mathcal{F}_{t,s}$  to contain the generator, it must contain the vertex  $(0, -3)$  and so has  $y$ -intercept  $-3$ . Thus,

$$\Upsilon_{-T(3,4)}(t) = -2(y_0 t/2) = -2(-3t/2) = 3t.$$

If  $t \in [2/3, 1]$ , the corresponding lines have slope  $m \geq -2$ , and the line of this slope with least  $y$ -intercept which bounds a lower half-space that contains a generator must contain the vertex  $(-1, -1)$ . This line has  $y$ -intercept  $y_0 = -1 + m = -1 + 1 - 2/t = -2/t$ . For these values of  $t$ ,

$$\Upsilon_{-T(3,4)}(t) = -2(y_0 t/2) = -2(-2/2) = 2.$$

This is consistent with the fact that  $\Upsilon_{-K}(t) = -\Upsilon_K(t)$ .

#### 4. DEFINING $\Upsilon_{K,t}^2$

We continue to work with a fixed basis  $\mathcal{B}$  so that chains are represented by a collection of vertices, but note that it is easily checked that the definitions are independent of the choice.

For a fixed value of  $t \in (0, 2)$ , recall we have by definition  $\gamma_K(t) = -\Upsilon_K(t)/2$ . Choose a  $\delta$  small enough to define  $p_t^-$  and  $p_t^+$ , as in Section 3.1, and let  $t^\pm = t \pm \delta$ . Let  $\mathcal{Z}^\pm$  be the set of cycles in  $\mathcal{F}_{t^\pm, \gamma_K(t^\pm)}$  which represent nontrivial elements in  $H_0(\text{CFK}^\infty(K))$ . Notice that each element of  $\mathcal{Z}^\pm$  is represented by a set of vertices that includes one at the lattice point  $p_t^\pm$ .

**Theorem 4.1.** *If  $\Delta\Upsilon'_K(t) > 0$ , then the sets  $\mathcal{Z}^-$  and  $\mathcal{Z}^+$  are disjoint.*

*Proof.* Let  $z^-$  be a cycle in  $\mathcal{Z}^-$ ; as noted above,  $z^-$  is represented by a set of basis elements that includes one at  $p_t^-$ . The hypothesis  $\Delta\Upsilon'_K(t) > 0$  combined with Theorem 3.3 implies that  $p_t^- \neq p_t^+$  and  $p_t^-$  is not contained in  $\mathcal{F}_{t^+, \gamma_K(t^+)}$ . So,  $z^-$  is not in  $\mathcal{Z}^+$ . Similarly, one can show the converse: no element of  $\mathcal{Z}^+$  belongs to  $\mathcal{Z}^-$ . This completes the proof.  $\square$

**Definition 4.2.** Let  $\mathcal{Z}^\pm = \{z_j^\pm\}$ . For each  $s \in [0, 2]$ , we set  $\gamma_{K,t}^2(s)$  to be the minimum value of  $r$  such that some  $z_i^-$  and  $z_k^+$  represent the same homology class in  $H_0(\mathcal{F}_{t, \gamma_K(t)} + \mathcal{F}_{s,r})$ .

Note that if  $\mathcal{Z}^-$  and  $\mathcal{Z}^+$  are not disjoint, then  $\gamma_{K,t}^2(s) = -\infty$ .

This definition is made more clear with an example.

**Example 4.3.** Figure 5 illustrates the complex for  $T(3, 4)$ . The half-space with boundary line containing the vertices  $(0, 3)$  and  $(1, 1)$  is shaded in. The slope of its support line is  $-2$ , and thus it corresponds to the value of  $\Upsilon_K$  at  $t = \frac{2}{1-(-2)} = \frac{2}{3}$ . Recall  $\Upsilon_K(\frac{2}{3}) = -2$ . In this case,  $\mathcal{Z}^\pm$  each contains exactly one element, represented by vertices at  $p^-$  and  $p^+$ , with coordinates  $(0, 3)$  and  $(1, 1)$ .

The second line in the figure, through  $(1, 3)$ , was chosen to have slope  $-\frac{1}{4}$ . The corresponding value of  $s$  is  $s = \frac{2}{1-(-1/4)} = \frac{8}{5}$ . The particular line drawn is the line of slope  $-\frac{1}{4}$  with the least  $j$ -intercept for which the classes represented by  $p^\pm$  are equal in  $H_0(\mathcal{F}_{t,u(t)} + \mathcal{F}_{s,r})$ . (This subcomplex is represented by the union of the two shaded half-spaces.) To compute the value of  $r$ , rather than find the  $j$ -intercept, we compute the value of  $\frac{s}{2}(j) + (1 - \frac{s}{2})(i)$  at  $(1, 3)$ . The value of  $r$  is given by

$$r = ((8/5)/2)(3) + ((1 - (8/5)/2)(1) = 13/5.$$

We now have that  $\gamma_{K, \frac{2}{3}}^2(\frac{8}{5}) = \frac{13}{5}$ .

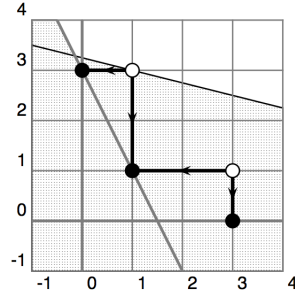


FIGURE 5.  $\text{CFK}^\infty(T(3, 4))$  doubly filtered

**Definition 4.4.**

$$\Upsilon_{K,t}^2(s) = -2\gamma_{K,t}^2(s) - \Upsilon_{K,t} = -2(\gamma_{K,t}^2(s) - \gamma_K(t)).$$

**Example 4.5.** For  $K = T(3, 4)$ ,  $\Upsilon_K(t)$  has a singularity at  $t = \frac{2}{3}$ ; at that singularity the slope has a positive jump. Note that  $\Upsilon_K(\frac{2}{3}) = -2$ . We compute that for  $s \in [0, 2]$  the line through  $(1, 3)$  gives

$$r = (s/2)(3) + (1 - s/2)(1) = 1 + s = \gamma_{K, \frac{2}{3}, s}^2.$$

It follows that

$$\Upsilon_{K, \frac{2}{3}}^2(s) = -2(1 + s) - (-2) = -2s.$$

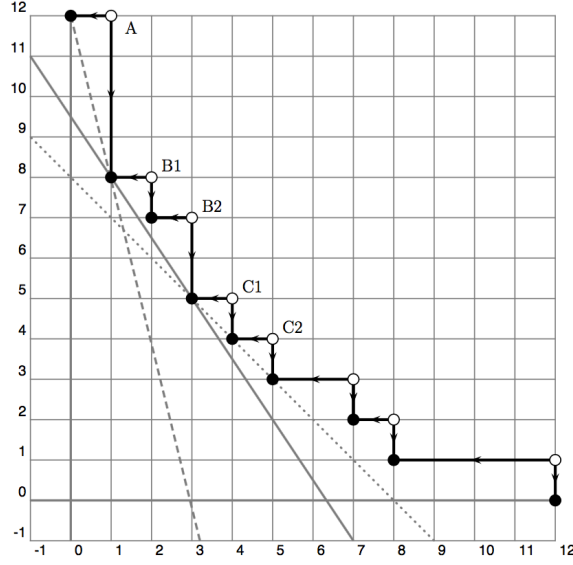
**Example 4.6.** A more interesting example is provided by the torus knot  $K = T(5, 7)$ , with  $\text{CFK}^\infty(K)$  as illustrated in Figure 6.

In the figure, support lines of slope  $m = -4, -3/2$ , and  $-1$  are drawn, and these correspond to the singular values of  $\Upsilon_K$  at  $t = 2/5, 4/5$  and  $1$ . A computation quickly shows that

$$\Upsilon_K(t) = \begin{cases} -12t & \text{if } 0 \leq t \leq 2/5 \\ -2 - 7t & \text{if } 2/5 \leq t \leq 4/5 \\ -6 - 2t & \text{if } 4/5 \leq t \leq 1. \end{cases}$$

We now consider  $\Upsilon_{K,t}^2$ : for  $t = 2/5$  (where the relevant point is labeled  $A$  in figure, at  $(1, 12)$ ; for  $t = 4/5$ , where the relevant points are labeled  $B1$  and  $B2$  at  $(2, 8)$  and  $(3, 7)$ ; and for  $t = 1$ , where the relevant points are labeled  $C1$  and  $C2$  at  $(4, 5)$  and  $(5, 4)$ . It is a straightforward computation to show that:

$$\Upsilon_{K, 2/5}^2(s) = \frac{14}{5} - 11s \text{ if } 0 \leq s \leq 2.$$

FIGURE 6.  $\text{CFK}^\infty(T(5,7))$ 

$$\Upsilon_{K,4/5}^2(s) = \begin{cases} \frac{8}{5} - 4s & \text{if } 0 \leq s \leq 1 \\ \frac{18}{5} - 6s & \text{if } 1 \leq s \leq 2. \end{cases}$$

$$\Upsilon_{K,1}^2(s) = \begin{cases} -2 + s & \text{if } 0 \leq s \leq 1 \\ -s & \text{if } 1 \leq s \leq 2. \end{cases}$$

**Example 4.7.** If  $\Delta\Upsilon'_K(t) > 0$ , then  $\Upsilon_{K,t}^2(s) < 0$  for some  $s$  by Theorem 4.1. In fact,  $\Upsilon_{K,t}^2(t) < 0$  in this case. If  $\Delta\Upsilon'_K(t) \leq 0$ , there are no known conditions on the signs of  $\Upsilon_{K,t}^2(s)$ . There are knots with  $\Delta\Upsilon'_K(t) < 0$  and  $\Upsilon_{K,t}^2(s) = \infty$ . For example,  $K$  can be the mirror image of  $T(3,4)$  or  $T(5,7)$ . Note that these knots have intersecting  $\mathcal{Z}^\pm$ .

On the other hand, there are  $\mathcal{K}$ -complexes  $C$  with  $\Delta\Upsilon'_C(t) < 0$  and  $\Upsilon_{C,t}^2(s) < 0$ . For example, consider the complex in Figure 7. For  $t = 1$ ,  $z^-$  is the sum of the generators at  $(-3, 1)$  and  $(0, -2)$ ;  $z^+$  is the sum of the generators at  $(-2, 0)$  and  $(1, -3)$ . This complex has  $\Delta\Upsilon'_C(1) = -4$  and  $\Upsilon_{C,1}^2(s) = -4$ .

**4.1. Concordance invariance.** The definitions for  $\Upsilon$  and  $\Upsilon^2$  can be extended to  $\mathcal{K}$ -complexes.

**Theorem 4.8.** *Adding acyclic summands to a  $\mathcal{K}$ -complex does not change the value of  $\Upsilon^2$ . Therefore,  $\Upsilon^2$  is a concordance invariant of knots.*

*Proof.* Let  $C$  be a  $\mathcal{K}$ -complex and  $A$  an acyclic complex. It is obvious that  $\gamma_{C \oplus A, t}^2(s) \leq \gamma_{C, t}^2(s)$  from the definition. To show the reversed inequality, let  $z^-$  and  $z^+$  be two cycles representing the same homology class in  $H_0$  as in Definition 4.2. Let  $c$  be a chain such that  $\partial c = z^- - z^+$ . We can write  $z^\pm = z_C^\pm + z_A^\pm$  and  $c = c_C + c_A$ , where  $z_C^\pm$  and  $c_C$  are elements of  $C$ , and  $z_A^\pm$  and  $c_A$  are elements of  $A$ . Then  $\partial c_C = z_C^- - z_C^+$  and  $\partial c_A = z_A^- - z_A^+$ . Since each  $z_C^\pm$  belongs to the corresponding subcomplex  $\mathcal{F}_{t^\pm, \gamma_C(t)}$  of  $C$ , this implies that  $\gamma_{C, t}^2(s) \leq \gamma_{C \oplus A, t}^2(s)$ , as desired. Finally, by Proposition 2.1 we see that  $\Upsilon^2$  is a concordance invariant of knots.  $\square$





Building from the previous example, one can compute that for the knot

$$K = T(4, 5) \# -T(2, 3; 2, 5) \# -T(2, 5)$$

the complex  $\text{CFK}^\infty(K)$  is as illustrated in Figure 9 (again, modulo acyclic summands). This computation calls on a change of basis as well as forming the tensor product. Given that this is the difference of two complexes with the same  $\Upsilon_K$  function, the difference has  $\Upsilon_K$  identically 0.

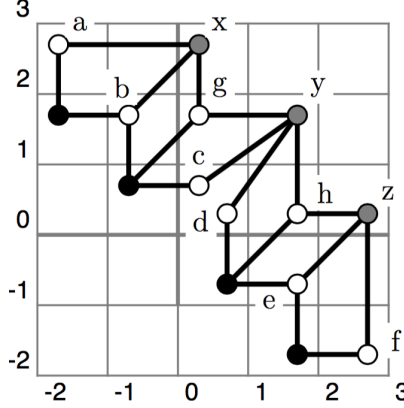


FIGURE 9.  $\text{CFK}(T(4, 5) \# -T(2, 3; 2, 5) \# -T(2, 5))$

Generators of grading 0 are drawn as white circles, those of grading  $-1$  are black, and those of grading 1 are gray. The group of cycles of grading 0 is generated by four cycles:

$$\langle a + b + c, d + e + f, g + c, h + d \rangle.$$

The pivoting vertices  $p^-$  and  $p^+$  are both  $(0, 0)$ . The cycles in  $\mathcal{Z}^\pm$  are unique:  $z^- = a + b + c$  and  $z^+ = d + e + f$ . The fact that these represent the same class in  $H_0(\text{CFK}^\infty(K))$  follows from their sum being the boundary of the chain  $x + y + z$ . This is the only possible chain whose boundary offers a relation between them.

To compute  $\Upsilon_{K,1}^2(s)$ , in the case that  $s \leq 1$  we see the support line must go through the vertex containing  $z$ , that is,  $(2, 0)$ . For this vertex, the value of  $\frac{s}{2}j + (1 - \frac{s}{2})i$  is  $2 - s$ . Multiplying by  $-2$  (and subtracting  $\Upsilon_K(1) = 0$ ) yields  $-4 + 2s$ .

For  $s \geq 0$ , the essential vertex is at  $(0, 2)$ , for which  $\frac{s}{2}j + (1 - \frac{s}{2})i$  has value  $s$ . Multiplying by  $-2$  gives  $-2s$ . Thus,

$$\Upsilon_{K,1}^2(s) = \begin{cases} -4 + 2s & \text{if } s \leq 1 \\ -2s & \text{if } s \geq 1. \end{cases}$$

Hom constructed this knot to build a concordance class that is nontrivial and which cannot be detected with  $\Upsilon_K$ , but can be detected by  $\epsilon$ . Here we see it is also detected by  $\Upsilon_{K,t}^2$ . In Section 7 we will use this example to build new ones related to bounds on the concordance genus.

## 5. SUBADDITIVITY

In the following statement of subadditivity, the appearance of the minimum instead of the maximum is explained by the presence of the factor of  $-2$  in the definition of  $\Upsilon^2$ . This will become clear in the first sentence of the proof. We state the theorem in terms of tensor products of  $\mathcal{K}$ -complexes, implying the similar statement for connected sums of knots.

**Theorem 5.1.** *For any  $\mathcal{K}$ -complexes  $C_1$  and  $C_2$ ,*

$$\Upsilon_{C_1 \otimes C_2, t}^2(t) \geq \min\{\Upsilon_{C_1, t}^2(t), \Upsilon_{C_2, t}^2(t)\}.$$

*Proof.* This is proved by showing that

$$(\gamma_{C_1 \# C_2, t}^2(t) - \gamma_{C_1 \# C_2, t}(t)) \leq \max\{(\gamma_{C_1, t}^2(t) - \gamma_{C_1, t}(t)), (\gamma_{C_2, t}^2(t) - \gamma_{C_2, t}(t))\}.$$

From the definition of  $\gamma^2$ , for  $r_1 = \gamma_{C_1, t}^2$  there is a chain  $c_1 \in \mathcal{F}_{t, \gamma_{C_1}(t)} + \mathcal{F}_{t, r_1}$  such that  $\partial c_1 = z_1^+ - z_1^-$ , with  $z_1^\pm$  as described in Section 4 for the complex  $C_1$ . Similarly, for the complex  $C_2$  there is a chain  $c_2$ . One has

$$\partial(c_1 \otimes z_2^+ + z_1^- \otimes c_2) = z_1^+ \otimes z_2^+ - z_1^- \otimes z_2^-.$$

The rest is arithmetic.  $\square$

Note that we consider  $\Upsilon_{K, t}(t)$  in the theorem. For  $\Upsilon_{K, t}(s)$  with  $t \neq s$ , the theorem would be false in general.

## 6. LINEAR INDEPENDENCE

In Figure 10 we illustrate a  $\mathcal{K}$ -complex which we denote  $K_n$ ; the sides of the square are of length  $2n$  and the central vertex is at the origin and has grading 0. The second diagram is the inverse complex, which we denote  $-K_n$ . A similar complex, in which the vertex  $B$  is not in the center of the square, was used in [14] as an example for which the nonvanishing of  $\Upsilon$  implies that the complex is not null concordant, whereas the  $\epsilon$ -invariant vanishes.

For  $K_n$  there are two cycles that represent generators of  $H_0$ :  $A + B$  and  $B + C$ , and these are homologous. For  $-K_n$  the classes are  $A + C$  and  $B$ , and these are homologous.

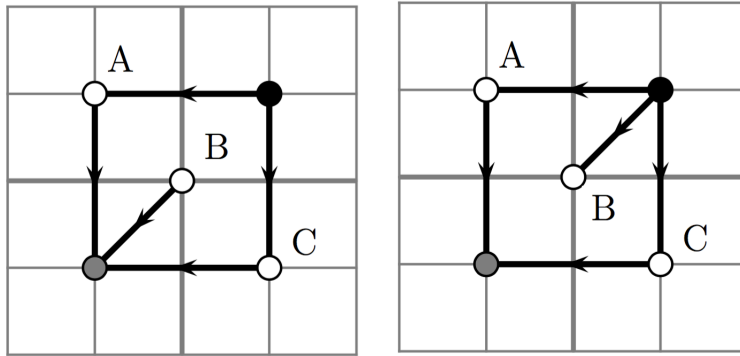


FIGURE 10.  $\mathcal{K}$ -complexes  $K_n$  and  $-K_n$

For the complex  $K_n$  one has  $\Upsilon_{K_n}(t) = 0$  for all  $t$ . We focus on  $\Upsilon_{K_n, 1}^2(1)$ , denoting this  $v^2(K)$ . The following is an easy exercise.

**Theorem 6.1.**  $v^2(K_n) = -2n$  and  $v^2(-K_n) = 0$ .

The linear independence in concordance of these complexes follows quickly.

**Theorem 6.2.** *If  $a_N > 0$ , then  $v^2(\oplus_{n=1}^N a_n K_n) = -2N$ .*

*Proof.* Write  $K = \oplus_{n=1}^N a_n K_n$ . By subadditivity,  $-2N \leq v^2(K) \leq 0$ . If there is a strict inequality,  $-2N < v^2(K)$ , then tensoring with copies of  $K_n$  or  $-K_n$  for  $n < N$  and applying subadditivity shows that  $-2N < v^2(a_N K_N)$ . Now tensor with  $-K_N$  (which has  $v^2 = 0$ ) repeatedly (more precisely,  $a_N - 1$  times) and apply subadditivity to find that  $-2N < v^2(K_N)$ , a contradiction.  $\square$

## 7. GENUS

Bounds on the three-genus of a knot  $K$ , and thus its concordance genus  $g_c(K)$ , based on  $\Upsilon_K(t)$  were presented in [12, 14]. These follow solely from the fact that for a knot of three-genus  $g$ , the filtration levels of all filtered basis elements  $x \in \text{CFK}^\infty(K)$  satisfy  $|\text{alg}(x) - \text{Alex}(x)| \leq g$ . (To be more precise, some complex representing the filtered chain homotopy class of  $\text{CFK}^\infty(K)$  has this property.) Thus, we have the following result corresponding to Theorem 10.1 of [12].

**Theorem 7.1.** *For every value of  $t$  and for all nonsingular points of  $\Upsilon_{K,t}^2(s)$  as a function of  $s$ ,  $|\Upsilon_{K,t}^2(s)| \leq g_c(K)$ . The jumps in  $\Upsilon_{K,t}^2(s)$  occur at rational numbers  $\frac{p}{q}$ . For  $p$  odd,  $q \leq g_c(K)$ . If  $p$  is even,  $\frac{q}{2} \leq g_c(K)$ .*

The complex  $\text{CFK}^\infty(T(3, 4))$  illustrated in Figure 1 is called a *stairway complex* of type  $[1, 2, 2, 1]$ . To set notation, its grading 0 generators will be denoted  $\{a_1, a_2, a_3\}$  and its grading 1 generators  $\{b_1, b_2\}$ . The  $a_i$  represent a generator of homology and are all homologous.

The complex for  $\text{CFK}^\infty(-T(3, 4))$  is denoted  $-[1, 2, 2, 1]$ . Its grading 0 generators are now still  $\{a_1, a_2, a_3\}$  but the set  $\{b_1, b_2\}$  represent grading  $-1$  generators. The set of 0-cycles is one dimensional, generated by  $a_1 + a_2 + a_3$ .

As illustrated in Figure 8,  $\text{CFK}^\infty(T(3, 4) \# -T(2, 3; 4, 5))$  and  $-\text{CFK}^\infty(T(2, 5))$  are, modulo acyclic complexes, both given by stairway complexes,  $[2, 2]$  and  $-[1, 1, 1, 1]$ .

**Example 7.2.** We now consider the connected sum of  $n$  copies of

$$K = T(3, 4) \# -T(2, 3; 4, 5) \# -T(2, 5),$$

denoted  $nK$ . We show  $g_c(nK) \geq 4n - 2$ . Modulo acyclic complexes,  $n(T(3, 4) \# -T(2, 3; 4, 5))$  is represented by the complex  $C_1 = [2, 2, \dots, 2]$  with a total of  $2n$  consecutive entries 2. Similarly, for the knot  $-nT(2, 5)$ , the complex is  $C_2 = -[1, 1, \dots, 1]$  with  $4n$  consecutive entries 1. (For details, see [5, Appendix B] or [6].)

We will denote the basis elements for  $C_1$  by  $A_i$  and  $B_i$ , and those for  $C_2$  by  $a_i$  and  $b_i$ . In the tensor product of these two complexes, there are only two types for which  $\text{alg}(\cdot) + \text{Alex}(\cdot) \leq 0$ . The first are of the form  $b_i \otimes A_j$ , with  $\text{alg}(\cdot) + \text{Alex}(\cdot) = -1$  and grading  $-1$ . The second are of the type  $a_i \otimes A_i$  for which  $\text{alg}(\cdot) + \text{Alex}(\cdot) = 0$  and grading 0.

Since  $\partial A_i = 0$ , it is easily seen that the only grading 0 cycles for which  $\text{alg}(\cdot) + \text{Alex}(\cdot) \leq 0$  are linear combinations of  $(\sum_{i=1}^{2n+1} a_i) \otimes A_j$ . It then follows that  $\mathcal{Z}^\pm$  each contain one element:  $z^- = (\sum_{i=1}^{2n+1} a_i) \otimes A_1$  and  $z^+ = (\sum_{i=1}^{2n+1} a_i) \otimes A_{n+1}$ .

Suppose that  $\partial w = z^- - z^+$ . Then clearly when  $w$  is expressed in terms of the given basis, both  $a_1 \otimes B_1$  and  $a_{2n+1} \otimes B_n$  appear. These are at filtration levels  $(-2n+2, 2n)$  and  $(2n, -2n+2)$ , respectively.

From these calculations, we have the following.

$$\Upsilon_{nK,1}^2(s) = \begin{cases} (4n-2)s - 4n & \text{if } s \leq 1 \\ (-4n+2)s + 4n - 4 & \text{if } s \geq 1. \end{cases}$$

As a corollary, we have  $g_c(nK) \geq 4n - 2$ .

Note that with care one can further show  $g_c(nK) \geq 4n$  since the boundary of  $a_1 \otimes B_1$  and  $a_{2n+1} \otimes B_n$  must appear in the complex. In [10] Hom asked whether another bound on the concordance genus, denoted there by  $\gamma$ , equals  $4n$  for these knots.

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